ON THE UNIQUENESS OF THE SOLUTIONS OF THE EQUATION OF WEAK CONVECTION IN THE STEADY STATE

(O EDINSTVENNOSTI RESHENII URAVNENIIA SLABOI STATSIONARNOI TEPLOVOI KONVEKTSII)

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In the present note we prove the uniqueness of the solutions to some problems involving heat transfer by weak convection in the absence of temperature discontinuities.

1. We consider the equation of weak convection in its non-dimensional form [1, 2]

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p - \nabla\times\nabla\times\mathbf{v} - \lambda\gamma\theta, \qquad \sigma\,\mathbf{v}\cdot\nabla\theta = \Delta\theta,\ \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

Here $\mathbf{v}$ denotes the velocity of the fluid, θ the difference between the temperature of the fluid and its mean value θ^*. The pressure is reckoned from its value $p^* = p(\theta^*)$ and in a state of mechanical equilibrium of the fluid. The symbols χ and σ denote, respectively, the Grosshof and Prandtl numbers, whereas γ denotes the unit vector in the direction of the gravitational acceleration.

We shall assume that the fluid occupies volume D within an arbitrary surface S in an infinitely rigid solid medium in which at an infinite distance from volume D a non-vertical temperature gradient is prescribed (here it is assumed that some characteristic length l of D is chosen as the unit of length). In this case (1.1) must be supplemented [3] with the equation which describes the temperature distribution θ in the solid

$$\Delta\theta^\circ = 0 \tag{1.2}$$

where θ^0 is also measured from θ^*.

Thus the set of equations (1.1), (1.2) is to be solved subject to the boundary conditions [3]

$$\mathbf{v} = 0, \text{ on } S \tag{1.3}$$

$$[\theta - \theta^\circ]_S = 0, \qquad \left[\frac{d\theta}{dn} - \alpha\,\frac{d\theta^\circ}{dn}\right]_S = 0 \tag{1.4}$$

$$|\nabla\theta^\circ| = \mathbf{a} \qquad \text{for } \rho\to\infty \tag{1.5}$$

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Here S denotes the boundary of the volume D, α is the ratio of the thermal conductivity κ^0 of the solid to that κ in the fluid; $\mathbf{a}$ denotes the unit vector in the direction of the temperature gradient at infinity, n is the normal to the surface S.

We shall seek solutions of the set (1.1) - (1.2) in the form of series [2]

$$\mathbf{v} = \lambda\mathbf{v}_1 + \lambda^2\mathbf{v}_2 + \cdots, \qquad \theta = \theta_0 + \lambda\theta_1 + \lambda^2\theta_2 + \cdots$$
$$p = \lambda p_1 + \lambda^2 p_2 + \cdots, \qquad \theta^\circ = \vartheta_0 + \lambda\vartheta_1 + \lambda^2\vartheta_2 + \cdots \tag{1.6}$$

Assuming that these series converge absolutely and uniformly, we substitute them into (1.1) - (1.2). In this manner we obtain sets of equations for the successive determination of the unknown functions in (1.6):

$$\Delta\theta_0 = 0, \qquad \Delta\vartheta_0 = 0, \qquad \nabla p_n + \nabla \times \nabla \times \mathbf{v}_n = \mathbf{F}_n \tag{1.7}$$

$$\Delta\theta_n = f_n, \qquad \Delta\vartheta_n = 0, \qquad \operatorname{div}\mathbf{v}_n = 0 \tag{1.8}$$

where

$$\mathbf{F}_n = -\gamma\theta_{n-1} - \sum_{j+k=n} (\mathbf{v}_j\nabla)\mathbf{v}_k \qquad (k, i = 1, \ldots, n) \tag{1.9}$$

$$f_n = \sigma \sum_{j+k=n} \mathbf{v}_j\nabla\theta_k \qquad \binom{j=1, \ldots, n}{k=0, 1, \ldots, n-1} \qquad (n = 1, 2, \ldots) \tag{1.10}$$

The function ϑ_0 can be represented in the form

$$\vartheta_0 = \mathbf{a}\cdot\rho + \vartheta_{01} \tag{1.11}$$

Hence, the functions $\vartheta_{01}, \vartheta_1, \ldots, \vartheta_n$, must be regular at infinity in accordance with (1.5).

Starting with (1.3) - (1.5) and taking into account (1.11) we obtain the following boundary conditions for (1.7), (1.8):

$$[\theta_0 - \mathbf{a}\cdot\rho - \vartheta_{01}]_S = 0$$

$$\left[\frac{d\theta_0}{dn} - \alpha\frac{d}{dn}(\mathbf{a}\cdot\rho + \vartheta_{01})\right]_S = 0$$

$$\nabla\vartheta_{01} = 0 \quad \text{for } \rho\to\infty, \quad \mathbf{v}_n|_S = 0 \quad \text{at } S \tag{1.12}$$

$$[\theta_n - \vartheta_n]_S = 0, \quad \left[\frac{d\theta_n}{dn} - \alpha\frac{d\vartheta_n}{dn}\right]_S = 0, \quad \nabla\vartheta_n = 0 \quad \text{for } \rho\to\infty \tag{1.13}$$

In this manner, in order to find each approximation $(\mathbf{v}_n, p_n, \theta_n, \vartheta_n)$ $(n = 1, 2, \ldots)$, it is necessary to solve a system of linear differential equations subject to the boundary conditions (1.13).

The problems of convergence of the methods of successive approximations have been considered by Linieikin [4] for the two-dimensional case, and by the Author* for the three-dimensional case.

* Stastisnarnaia teplovaia konvektsiia v trubakh peremiennovo Getcheniia (Steady-state heat transfer by convection in channels of variable cross-section). *Thesis, Perm University, 1954.*

The problem of the uniqueness of the solutions of the equations of heat transfer by convections when the conditions of heating are prescribed directly on the boundary of the volume in the forms of a rectangular parallelepiped have been considered by Veniamienovitch [5].

In what follows, we shall prove the uniqueness of the expansions in (1.6) for the problem as formulated in the preceding discussion.

2. Uniqueness of the expansions in (1.6). We assume that the approximations $(\mathbf{v}_k, \nabla p_k, \theta_k, \vartheta_k)$ $(k = 1, 2, \ldots, n-1)$ have been determined up to order $(n-1)$, and that each of them proves to be unique. We then prove that the n'th approximation $(\mathbf{v}_n, \nabla p_n, \theta_n, \vartheta_n)$ is also unique.

We now admit the opposite: the system of equations (1.7), (1.8) possess two different solutions $\mathbf{v}_n$ ∇p_n, θ_n, ϑ_n and $\mathbf{v}_n'$, $\nabla p_n'$, θ_n', ϑ_n', which satisfy the conditions (1.13).

We introduce the notation

$$\mathbf{u} = \mathbf{v}_n - \mathbf{v}_n', \qquad \nabla\pi = \nabla p_n - \nabla p_n', \qquad t = \theta_n - \theta_n', \qquad \vartheta = \vartheta_n - \vartheta_n' \tag{2.1}$$

The latter functions satisfy the linear equations

$$\nabla\pi + \nabla \times \nabla \times \mathbf{u} = 0, \quad \triangle t = 0 \tag{2.2}$$

$$\triangle\vartheta = 0, \quad \operatorname{div} \mathbf{u} = 0 \tag{2.3}$$

and the boundary conditions

$$\mathbf{u} = 0, \qquad t - \vartheta = 0, \qquad \frac{dt}{dn} - \alpha\frac{d\vartheta}{dn} = 0 \qquad \text{on } S' \tag{2.4}$$

$$\nabla\vartheta = 0 \qquad \text{for} \qquad \rho \to \infty \tag{2.5}$$

We multiply the first equation (2.2) by $\mathbf{u}$, and the first equation (2.3) by t and integrate over the volume D; we then obtain

$$\int_D \mathbf{u}\nabla\pi \, dD + \int_D (\mathbf{u}\cdot\nabla\times\nabla\times\mathbf{u})\, dD = 0, \qquad \int_D t\Delta t\, dD = 0 \tag{2.6}$$

Making use of the second equation (2.3) and of the first condition (2.4), we obtain from the first equation (2.6) that

$$\int_D (\nabla\times\mathbf{u})^2\, dD = 0 \qquad \text{or} \qquad \nabla\times\mathbf{u} \equiv 0 \qquad \text{inside } D \tag{2.7}$$

In view of the second equation (2.3) and the first equation (2.4), it follows that

$$\mathbf{u} = \mathbf{v}_n - \mathbf{v}_n' \equiv 0 \qquad \text{inside } D \tag{2.8}$$

Further, taking into account (2.8), we obtain from the first equation (2.2) that

$$\nabla\pi = \nabla p_n - \nabla p_n' \equiv 0 \qquad \text{inside } D \tag{2.9}$$

Making use of Green's formula, we represent the second integral in (2.6) in the form

$$\int_D (\nabla t)^2\, dD - \int_S t\,\frac{dt}{dn}\, dS = 0 \tag{2.10}$$

We now transform the second integral in (2.10) making use of the second and third equation (2.4); we obtain

$$\int_S t\,\frac{dt}{dn}\, dS = -\alpha \int_S \vartheta\,\frac{d\vartheta}{d\nu}\, dS \qquad \left(\frac{d}{dn} = -\frac{d}{d\nu}\right) \tag{2.11}$$

Hence ν denotes the inner normal to the surface S, the function ϑ, which satisfied the first condition (2.3) is harmonic outside D and is regular at infinity in accordance with (2.5). Hence the left-hand part of (2.11) can be transformed to

$$\int_S t\,\frac{dt}{dn}\, dS = -\alpha \int_{D^\circ} (\nabla\vartheta)^2\, dD^\circ \tag{2.12}$$

where D^0 is the κ space exterior to D.

Taking into account (2.11) and (2.12), equation (2.10) can be transformed to

$$\int_D (\nabla t)^2\, dD = -\alpha \int_{D^\circ} (\nabla\vartheta)^2\, dD^\circ \tag{2.13}$$

Since $\alpha > 0$, it follows from (2.13) that

$$\nabla t \equiv 0 \quad \text{inside } D, \qquad \nabla\vartheta \equiv 0 \quad \text{inside } D^0$$

Hence, in view of (2.4) and (2.5), we have

$$t = \theta_n - \theta_n' \equiv 0 \quad \text{inside } D, \qquad \vartheta = \vartheta_n - \vartheta_n' = 0 \quad \text{inside } D^\circ \tag{2.14}$$

The relations (2.8), (2.14) show that if all approximations up to order $(n-1)$ are unique, then the n'th approximation is also unique. It is easy to verify that the zeroth approximation is unique. Hence, in accordance with what has been proved, all approximations $(v_n, p_n, \theta_n, \vartheta_n)$ are unique. It follows that the expansions (1.6) which solve the system (1.1) - (1.2) subject to the boundary conditions (1.3) - (1.5) are unique for those values of the Grasshof number λ, for which the series (1.6) converges absolutely and uniformly. Under those conditions, according to (2.9), the presence p is determined up to a constant; this is usually sufficient in problems of natural convection.

The uniqueness of the expansion in terms of the Grasshof number of solutions of the system of equations of heat convection in a volume of arbitrary shape for problems without discontinuities and with a prescirbed temperature distribution or heat flux on the boundary, follows directly from what has been proved above.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, E.M., *Mekhanika sploshnykh sred (Mechanics of continuous media)*. Gostekhizdat, 1953.

2. Shaposhnikov, I.G., K teorii slaboi konvektsii (On the theory of weak convection). *Zhur. tekh. fiz.* Vol. 22, p. 826, 1952.

3. Ostroumov, G.A., *Svobodnaia konvektsiia v usloviakh vnutrennei zadachi (Free convection under conditions giving rise to an internal problem)*. Gostekhizdat, 1952.

4. Linieikin, P.S., Ob uravnieniiakh teplovoi konvektsii (On the equations of heat transfer by convection). *PMM* Vol. 15, No. 4, p. 433, 1951.

5. Veniamienov, V.V., Rasprostranenie tepla v sosude s zhidkost'iu (The transfer of heat in a vessel containing a fluid). *Zhur. Geofiz.* Vol. 3, No. 1, p. 42, 1933.

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